cision on the IBM 7094 show the appearance of instabilities for Tollmein-Schlichting waves with longer wavelengths when $\alpha < 0.1$. Further calculations in the low Reynolds number and low α region are required.

The effect of wall tension on the three modes of instability for a particular wavelength of the disturbance and Reynolds number is shown in Fig. 3. The disturbance is stable when the imaginary part of c is positive and unstable when it is negative. $C_{\rm I}$ is the Tollmein-Schlichting mode modified by flexibility. $C_{\rm II}$ is a stable wave which propagates in the upstream direction. The third eigenvalue $C_{\rm III}$ is unstable and the eigenfunction becomes unbounded as $c \to 1$. This can usually be cured by addition of wall damping as shown for $K_3 = 1500$. Damping is destabilizing for $C_{\rm I}$, but to a lesser extent, so that it remains stable Additional calculations are required to obtain neutral stability curves for $C_{\rm II}$ and $C_{\rm III}$ as well as for all the eigenvalues in the region where $\alpha < 0.1$. These results will be presented in the near future.

References

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Design of Rectangular Panels with Biaxial Stresses

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Nomenclature

- a = length of rectangular panel
- b = width of rectangular panel
- C_{mn} = amplitude of deflection of mn mode
- m = number of half-waves in longitudinal (x) direction
- n = number of half-waves in transverse (y) direction
- t = panel thickness
- w = lateral deflection
- x = longitudinal direction
- y = transverse direction
- C_τ = stability constant for panel in pure shear
- D = bending stiffness of panel $[Et^3/12(1-\nu^2)]$
- E = effective modulus of elasticity
- K_c = equivalent stability constant for biaxial compression
- N_x = compression loading per inch acting in x direction
- N_y = compression loading per inch acting in y direction
- N_{xy} = shear loading per inch
- $\bar{N}_x$ = equivalent uniaxial compression loading in x direction for a panel loaded in biaxial compression and shear
- γ = ratio of compression loadings (N_y/N_x)
- ξ = nondimensional parameter (m^2b^2/n^2a^2)
- σ_x = stress in x direction
- σ_y = stress in y direction
- $\bar{\sigma}$ = characteristic stress function (π^2D/b^2t)

THE analysis of a simply supported rectangular panel subjected to biaxial-compression is treated in paragraph 64 of Ref. 1. The analysis determines the axial stress σ_y that exists when the panel buckles because of a known axial stress σ_x. The formulation is not convenient, however, for designing a panel when only the loads (N_x and N_y) are known. The usual interaction type of equation, which treats the over-all stability as functions of the stability of independent modes, can only be an approximation since the buckle pattern will change with the aspect and load ratios. The technique recommended is to use the standard design technique for a uniaxially loaded plate with an equivalent stability constant that is a function of the aspect and load ratios.

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The equilibrium equation for the rectangular plate with biaxial loading is

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} = 0 \quad (1)$$

Substituting the general eigenvector

$$w = C_{mn}\sin(m\pi x/a)\sin(n\pi y/b) \quad (2)$$

which satisfies the simply supported boundary conditions, into Eq. (1) results in the following relationship which is satisfied by the critical loading:

$$N_x\left(\frac{m^2\pi^2}{a^2}\right) + N_y\left(\frac{n^2\pi^2}{b^2}\right) = D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 \quad (3a)$$

Let

$$\bar{\sigma} = \pi^2D/b^2t \quad (3b)$$

$$\xi = m^2b^2/n^2a^2 \quad (3c)$$

$$a > b \qquad N_x/t = \sigma_x > \sigma_y = N_y/t \quad \text{(compression positive)} \quad (3d)$$

$$\gamma = N_y/N_x = \sigma_y/\sigma_x \leq 1 \quad \text{(can be negative)} \quad (3e)$$

Substitute the preceding in Eq. (3a) to determine the minimum value of σ_x satisfying the buckling condition. This results in

$$\sigma_x = \min[\bar{\sigma}n^2(1+\xi)^2/(\gamma+\xi)] \quad (3f)$$

where m and n are integers.

The minimum value of σ_x corresponds to $n = 1$ (σ_x is monotonic with n) and for the smallest integer value of m which is greater than $(a/b)(1-2\gamma)^{1/2}$

The equation

$$m \geq (a/b)(1-2\gamma)^{1/2} \qquad (\gamma \leq \tfrac{1}{2}) \quad (4a)$$

is obtained by setting $\partial\sigma_x/\partial\xi = 0$ and solving for m when $n = 1$. For values of γ greater than $\frac{1}{2}$, the value of σ_x is monotonic with respect to ξ and the minimum value occurs at the smallest value of ξ. This corresponds to

$$m = 1 \qquad (\gamma \geq \tfrac{1}{2}) \quad (4b)$$

The design of a simply supported panel can be obtained, therefore, as follows:

1) For the given aspect ratio $a/b \geq 1$ and load ratio $N_y/N_x = \gamma \leq 1$, determine the smallest integer $m \geq (a/b)(1-2\gamma)^{1/2}$ (for $\gamma > \frac{1}{2}$, $m = 1$).

2) Determine $\xi = m^2b^2/a^2$.

3) Establish equivalent uniaxial stress equation

$$\sigma_x = \frac{(1+\xi)^2}{\gamma+\xi}\,\frac{\pi^2}{12(1-\nu^2)}\,E\left(\frac{t}{b}\right)^2 = K_cE\left(\frac{t}{b}\right)^2 \quad (5a)$$

with

$$K_c = \frac{(1+\xi)^2}{\gamma+\xi}\,\frac{\pi^2}{12(1-\nu^2)} \quad (5b)$$

4) Design plate with K_c rather than $[(mb/a) + (a/mb)]^2[\pi^2/12(1-\nu^2)] \to 3.62$ which corresponds to the uniaxial compression case. For the elastic case the design equation becomes

$$t = (N_xb^2/K_cE)^{1/3} \quad (6)$$

For plastic design an effective modulus must be defined and the plate designed by a graphical method (e.g., see subsection IIC 1 of Ref. 2).

5) For combined shear and biaxial edge loadings a formulation based upon the interaction formulation

$$(\sigma_x/\sigma_{x_{cr}}) + (\tau_{xy}/\tau_{xy_{cr}})^2 = 1 \quad (7a)$$

is recommended since the eigenvectors for biaxial and shear loadings are different.

Using the design technique presented in subsection IIF 1 of Ref. 2 results in designing the panel for an equivalent loading

$$\bar{N}_x = N_x[1 + \{2(N_{xy}/N_x)(K_c/C_\tau)\}^2]^{1/2}/2(N_{xy}/N_x)(K_c/C_\tau) \quad (7b)$$

where

$$C_\tau \sim [\pi^2/12(1 - \nu^2)](5.34 + 4b^2/a^2) \quad (7c)$$

is the stability constant for the simply supported plate and N_{xy} is the shear loading.

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Ablation of a Hollow Sphere

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Introduction

Citron[1] and Goodman[2] have presented approximate techniques for the solutions of ablating slabs. In this note, the two methods are adopted for determining the ablation characteristics of a hollow sphere and the results are compared.

Theoretical Analysis

A sphere with an initial outside radius a is at a constant initial temperature T_i. The inner surface, $r = b$, is insulated. The sphere is subjected to a point-symmetric, radial heat flux $Q(t)$ acting at its ablating outer surface whose radius is $r = r_s(t)$.

In the analysis, it is assumed that the heated surface remains at the melt temperature T_m, and that the molten material is immediately removed upon formation.

The premelt analysis, which is applicable during the period beginning with the initial time $t = 0$ and ending with the melt time $t = t_m$, which is the time at which the melt temperature T_m is first reached at the heated surface, has been adequately treated in the literature[3,4] and will not be pursued here.

The point-symmetric heat-conduction equation in spherical coordinates is

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2 \frac{\partial T}{\partial r}\right) \qquad b < r < r_s(t) \quad (1)$$

and the initial and boundary conditions are as follows:

$$T(r, t_m) = T_0(r, t_m) \quad (2a)$$

$$r_s(t_m) = a \quad (2b)$$

$$T(r_s, t) = T_m \quad (2c)$$

$$Q(t) = k(T_m)(\partial T/\partial r)_{r_s,t} - \rho L(dr_s/dt) \quad (2d)$$

$$(\partial T/\partial r)_{b,t} = 0 \quad (2e)$$

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where ρ, c, k, and L denote the density, specific heat, thermal conductivity, and latent heat of fusion of the material, and T_0 is the premelt temperature distribution. An auxiliary condition, $(dr_s/dt)_{t=t_m} = 0$, can be imposed from a consideration of continuity of heat input at the melt time.[1]

The solution of the posed problem will now be treated by two different approximate numerical techniques.

Method 1

In this approach, the technique utilized by Citron[1] for a slab is applied to the spherical shell. The method consists of applying a transformation which allows the consideration of a body of constant unit thickness at all times in lieu of a body of varying thickness, and then expressing the temperature distribution at any time in this unit body by a Taylor series expansion in space about the melting surface.

Using the transformation $Z = (r - b)/(r_s - b)$, and defining the following nondimensional parameters,

$$\tau = \frac{\kappa(T_m)(t - t_m)}{(a - b)^2} \qquad R_s(\tau) = \frac{a - r_s}{a - b}$$

$$\theta(Z, \tau) = \frac{T - T_i}{T_m - T_i} \qquad \bar{Q}(\tau) = \frac{Q(t)}{Q_0}$$

$$\bar{k} = \frac{k(T)}{k(T_m)} \qquad \bar{c} = \frac{c(T)}{c(T_m)}$$

$$B^* = \frac{k(T_m)[T_m - T_i]}{Q_0[a - b]} \qquad M^* = \frac{c(T_m)[T_m - T_i]}{L}$$

where Q_0 is the heat input at $t = t_m$ and κ is the thermal diffusivity, we obtain the transformed heat-conduction equation (1), which is valid for temperature-dependent material properties

$$\frac{\partial^2\theta}{\partial Z^2} = \frac{\bar{c}}{\bar{k}}(1 - R_s)^2\frac{\partial\theta}{\partial\tau} + \frac{\bar{c}}{\bar{k}}Z(1 - R_s)\dot{R}_s\frac{\partial\theta}{\partial Z} - \frac{2(1 - R_s)}{(1 - R_s)Z + \dfrac{b/a}{1 - b/a}}\frac{\partial\theta}{\partial Z} - \frac{1}{\bar{k}}\frac{d\bar{k}}{d\theta}\left(\frac{\partial\theta}{\partial Z}\right)^2 \quad (3)$$

Differentiation with respect to τ is denoted by ($\dot{}$). Conditions (2a–2e) become

$$\theta(Z,0) = \theta_0(Z, 0) \quad (4a)$$

$$R_s(0) = 0 \quad (4b)$$

$$\theta(1, \tau) = 1 \quad (4c)$$

$$(\partial\theta/\partial Z)_{1,\tau} = (1 - R_s)[(\bar{Q}/B^*) - (\dot{R}_s/M^*)] \quad (4d)$$

$$(\partial\theta/\partial Z)_{0,\tau} = 0 \quad (4e)$$

and $(dr_s/dt)_{t=t_m} = 0$ transforms to $\dot{R}_s(0) = 0$.

The following is a brief review of Citron's procedure: One assumes that $\theta(Z, \tau)$ can be expressed as a Taylor series expansion in space about the melting face $Z = 1$, i.e.,

$$\theta(Z, \tau) = \theta(1, \tau) + (\partial\theta/\partial Z)_{1,\tau}(Z - 1) + (\partial^2\theta/\partial Z^2)_{1,\tau}[(Z - 1)^2/2!] + \ldots \quad (5)$$

The application of condition (4e) to Eq. (5) yields

$$0 = \left(\frac{\partial\theta}{\partial Z}\right)_{1,\tau} - \left(\frac{\partial^2\theta}{\partial Z^2}\right)_{1,\tau} + \frac{1}{2!}\left(\frac{\partial^3\theta}{\partial Z^3}\right)_{1,\tau} - \ldots \quad (6)$$

where $(\partial\theta/\partial Z)_{1,\tau}$, $(\partial^2\theta/\partial Z^2)_{1,\tau}$, . . ., $(\partial^n\theta/\partial Z^n)_{1,\tau}$ can all be expressed in terms of R_s, $\dot{R}_s$, $\ddot{R}_s$, $\dddot{R}_s$, etc., by utilizing conditions (4c) and (4d), and by successively differentiating Eq. (3) with respect to Z, and evaluating the resulting expression at $Z = 1$. The substitution of these derivatives into Eq. (6) yields a nonlinear, ordinary differential equation involving R_s and its derivatives. It should be noted that this equation